III

The Lorentz Transformation Equations

Using the new concepts of time and space introduced in this paper, I propose to demonstrate quantitatively the following well known consequences of the Lorentz transformation equations:

(1) Time dilation
(2) Length contraction
(3) Phase difference in the synchronization of clocks

With these results, we can then construct the Lorentz transformation equations:

$$t' = \frac{t - vx/c^2}{(1 - v^2/c^2)^{1/2}},$$

$$x' = \frac{x - vt}{(1 - v^2/c^2)^{1/2}}, \qquad y' = y, \qquad z' = z.$$

where v is along the x-axis.

For all the following discussions in this section, we would be considering two inertial frames of reference: S, which would be considered as stationary, and S′, which would be moving along the x-axis with velocity v relative to S. We shall denote t, x, y, z and t′, x′, y′, z′ as the time and positional coordinates of S and S′ respectively.

Time Dilation

We are here comparing the time reading of events A and B which are positionally correlated according to one inertial reference frame S′ (ie. its proper time) with the time reading by another inertial reference frame S, which does not consider the same events to be positionally correlated.

Consider Fig. 1 which shows events A and B as seen by S. A light ray has been sent forwards and backwards along an axis perpendicular to the line of motion, linking A and B.

From man's arbitrary definition of time, the distance the light ray travels in his demarcated space between any two events is directly the measure of his considered elapsed time between the two events. If we assign the distance between events A and C according to S to be 1, the distance between A and C according to S′ would be $(1 - v^2/c^2)^{1/2}$. Hence it follows that the time between A and B as noted by S would be greater than that noted by S′ and would be given by the equation:

$$\Delta t = \frac{\Delta t'}{(1 - v^2/c^2)^{1/2}}$$

Hence we have the time dilation formula. It should be noted that this is also directly a consequence of how we define simultaneity between positionally uncorrelated events.

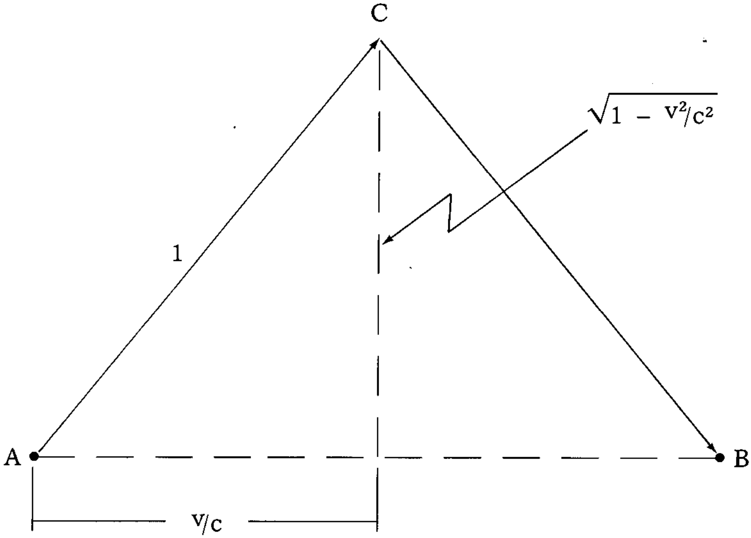

Fig. 1. The diagram shows the path of a light ray as seen by an inertial reference frame S which considers events A and B to be positionally uncorrelated. The same light ray, as seen by another inertial reference frame S′ which considers A and B to be positionally correlated, would have traversed a distance smaller by a factor of $(1 - v^2/c^2)^{1/2}$. Hence, according to S′, the elapsed time between A and B would be smaller by $(1 - v^2/c^2)^{1/2}$ when compared to that according to S.

Length Contraction

We are now comparing the distance between two events A and B as determined by two inertial reference systems. The events will be considered as simultaneous in one frame of reference and not in the other.

Fig. 2 shows events A and B as seen by reference frame S where they are not considered to be simultaneous. Reference frame S′ which is moving relative to S, considers A and B to be simultaneous. x_1 and x_2 are events which are positionally correlated according to S′. Light rays are sent from event x_1 to events A, B, C and D, and then on to event x_2 (ie. they all arrive at the same time). Fig. 2 shows the paths taken by these light rays as seen by S.

According to man's arbitrary definition of time and space, events A, B, C and D are considered by S′ to be of equal distance from x_1 and also to be simultaneous to each other. According to S, the paths taken by the light rays are of equal distances since they all originate and end at the same two events. Hence:

$$x_1Ax_2 = x_1Bx_2 = x_1Cx_2 = x_1Dx_2$$

If, according to S, we set distance $x_1C = 1$, then $AB = 2$ and $CD = 2(1 - v^2/c^2)^{1/2}$. Hence the ratio of AB to CD will be:

$$\frac{AB}{CD} = \frac{1}{(1 - v^2/c^2)^{1/2}}$$

As CD is perpendicular to the line of motion, its distance is equivalent in both frames of reference. According to S′, events C and D and events A and B are separated by equal distances, which means that in reference frame S, comparing AB with CD is equivalent to comparing Δx with $\Delta x'$ (where Δx is the proper length along the x-axis on reference frame S and $\Delta x'$ is the corresponding distance according to S′). Therefore:

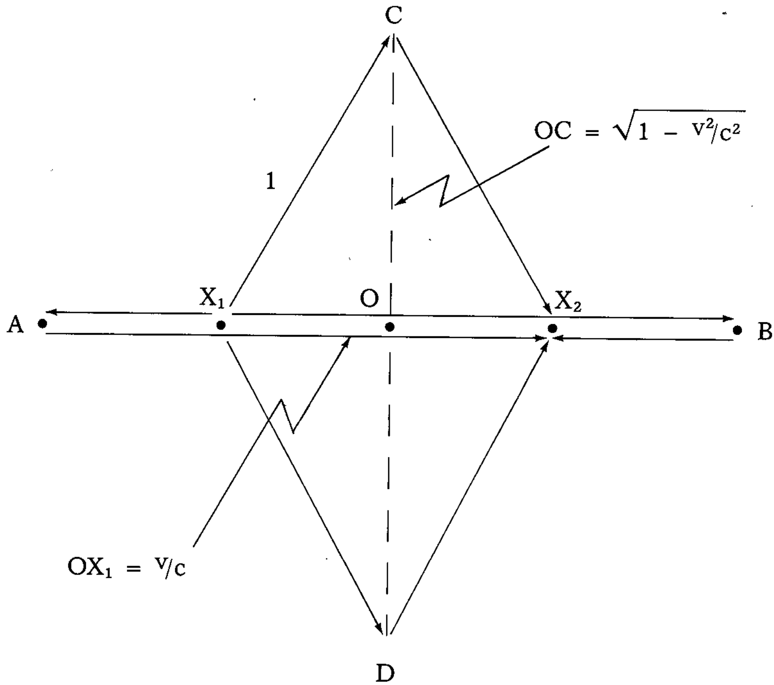

Fig. 2. The diagram shows the paths of light rays leaving from event x_1 and returning to event x_2 as seen by inertial reference frame S. x_1 and x_2 are events considered to be positionally correlated by another inertial reference frame S′ moving with velocity v relative to S. The distance AB is considered longer than CD according to S, but is considered the same according to S′. As the distance CD is equal for both frames of reference, there is, then, a "length contraction effect" for the distance between events A and B, when determined by the two different inertial systems.

$$AB = \frac{CD}{(1 - v^2/c^2)^{1/2}} \quad \Rightarrow \quad \Delta x = \frac{\Delta x'}{(1 - v^2/c^2)^{1/2}}$$

Hence, we have the length contraction formula. It should be noted that according to S, events A and B are not simultaneous and comparison of length is only possible because of the arbitrary method of linking events by positional correlation.

Phase Difference in the Synchronization of Clocks

We are now going to demonstrate quantitatively the reason for the phase difference term:

$$\frac{-\ vx/c^2}{(1 - v^2/c^2)^{1/2}} \quad \text{in the equation:} \quad t' = \frac{t - vx/c^2}{(1 - v^2/c^2)^{1/2}}$$

The presence of the term means that clocks further away in the positive x direction on the moving reference frame S′ will show earlier times.

Fig. 3a shows the paths of light rays originating at event O and travelling to events A and B as seen by stationary reference frame S. Fig. 3b shows the paths of the same light rays according to S′. The distances traversed by the light rays from O to A and from O to B are the same (= ½) according to S, and so A and B are considered as simultaneous events. Also, according to S, the reference frame S′ has moved a distance ½v/c during the time the light ray took to travel from O to A (or from O to B). The observer on S′ would say therefore that the distances traversed by the two light rays are different. For him the distances would be O′A and O′B where O′ is the event considered by S′ to be positionally correlated to event O, and considered by S to be simultaneous to A and B.